AERODYNAMIC CHARACTERISTICS OF THIN WEDGE-SHAPED PROFILES IN TRANSONIC FLOWS

(AERODINAMICHESKIE KHARAKTERISTIKI TONKOGO KLINOVIDNOGO PROFILIA V ZVUKOVOM POTOKE)

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Consider the problem of calculating the aerodynamic characteristics of a thin wedge-shaped profile placed at a small angle of attack in a sonic flow. When the solution is known in the subsonic region it is necessary to continue it past the dividing characteristic line. In the supersonic region of the flow it is generally possible to utilize the method of characteristics as has for instance been done in [1]. However, in nearly sonic flows it is possible to devise analytical methods for this purpose. In [2] such a method was based on the hypothesis that the governing equation of Euler-Darboux can be replaced by the wave equation, but this is inconvenient for the continuation of the field over the upper surface of the profile.

Apparently, the continuation of the solution was similarly constructed (without indication of underlying simplifications) in [3], where results of computations of pressure distributions over the upper surface of a flat plate are presented.

Below, the continuation of the solution past the dividing characteristic is constructed starting from the solution of a certain boundary-value problem of the Euler-Darboux equation, which takes on the assigned values on the dividing characteristic and the zero value on the straight line corresponding to the contour of the profile. It is noted that reference [4] solved a similar problem of the Euler-Darboux equation, but for different, simpler boundary values.

As part of the results the complete picture of the pressure distribution over the wedge-shaped profile is obtained as a function of the angle of attack. Also, the limits of applicability of the investigated patterns of flow are shown.

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1. Problem of flow functions in the hodograph plane

For simplicity, consider a thin symmetric double-wedge airfoil with half nose-angle θ_0, chord length b, and angle of attack α. As in [5], the profile is studied by means of the following flow patterns.

I. The stagnation point is assumed to occur at the leading edge. The flow is analogous to that at zero angle of attack.

II. The stagnation point is assumed to occur on the lower front face of the wedge. The flow over upper and lower front faces of the profile is analogous to the flow over upper and lower surfaces of a flat plate.

III. The flow over the complete profile is analogous to the flow around a flat plate.

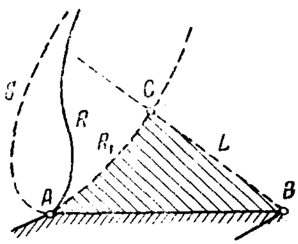

Fig. 1.

Fig. 1 shows the flow over the supersonic face of the profile, where the sonic speed is reached at the point A. The solution is continued into the region past the dividing characteristic R. The flow accelerates around the point A from the sonic curve S to the dividing characteristic R and then to the [terminal] characteristic R_1. Defining R_1 as belonging to the first family of characteristics, let us construct the characteristic L of the second family, which passes through the trailing edge B. This defines the region ABC. The solution in this region is found from the conditions on the characteristic R_1 and the surface AB.

In actuality, downstream of R there may arise one or more shock waves so that the above description of the flow field may be disturbed. However, visual studies of flow over such profiles indicate that, for small angles of attack (Flow pattern I), such shock waves are weak and may be neglected. For larger angles of attack (Flow patterns II and III), the discontinuities over the top surface are oblique so that the regime of supersonic flow over the upper front [sic] face of the profile is preserved. Also their intensity is smaller, the smaller θ_0 and α. Therefore, in the first approximation we shall assume that the flow is continuous downstream of the characteristic R.

For our variables we choose

$$\theta = a\left(\frac{v_y}{a_* \tau} - l\right), \qquad \eta = a^{2/3}\,\frac{(\varkappa + 1)^{1/3}}{\tau^{2/3}}\left(1 - \frac{v_x}{a_*}\right)$$

For the unknowns we take

$$\psi(\theta,\eta)=\frac{\tau^{1/3}(\varkappa+1)^{2/3}}{a^{2/3}l_*}y, \qquad \varphi(\theta,\eta)=\frac{x}{l_*}$$

where

$$a=1, \quad l=-\frac{\alpha+\theta_0}{2\theta_0}, \quad \tau=2\theta_0, \quad l_*=\frac{b}{2} \quad \text{(Flow pattern I)}$$

$$a=\frac{2\theta_0}{\pi}, \quad l=-\frac{\alpha+\theta_0}{2\theta_0}, \quad \tau=2\theta_0, \quad l_*=\frac{b}{2} \quad \text{(Flow pattern II)}$$

$$a=\frac{\alpha}{\pi}, \quad l=-1, \quad \tau=\alpha, \quad l_*=b \quad \text{(Flow pattern III)}$$

Here, x, y represent the cartesian coordinates of the two-dimensional flow field and v_x, v_y the components of the velocity. We note that, for nearly sonic flows, up to terms of higher order and a scalar factor, ψ is the stream function and ϕ the velocity potential.

The equations of motion in the hodograph plane take the form [6]:

$$\frac{\partial\psi}{\partial\eta}+\frac{\partial\varphi}{\partial\theta}=0, \qquad \frac{\partial\varphi}{\partial\eta}-\eta\frac{\partial\psi}{\partial\theta}=0 \tag{1.1}$$

or, with ϕ eliminated:

$$\frac{\partial^2\psi}{\partial\eta^2}+\eta\frac{\partial^2\psi}{\partial\theta^2}=0 \tag{1.2}$$

The characteristics of equation (1.2) in the hyperbolic region are expressible as

$$\xi=\theta-\frac{2}{3}(-\eta)^{3/2}, \qquad \lambda=\theta+\frac{2}{3}(-\eta)^{3/2} \tag{1.3}$$

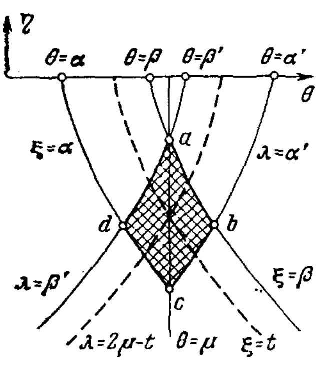

Fig. 2.

In the plane $\theta\eta$, the problem of smooth continuation from the subsonic field to the supersonic one is reducible to the following two cases.

1. To find the solution $u(\theta,\eta)$ of equation (1.2) in the strip

$$\alpha \leqslant \xi \leqslant \beta, \quad \alpha \leqslant \lambda \leqslant \alpha', \quad \eta \leqslant 0 \quad (\beta' \leqslant \lambda \leqslant \beta, \quad \alpha \leqslant \xi \leqslant \alpha', \quad \eta \leqslant 0)$$

which reduces to zero on the characteristic $\xi = \alpha (\eta = \alpha')$ and takes on the prescribed value $\nu(\theta) = [\partial\psi / \partial\eta]_{\eta=0}$ on the dividing (parabolic) line (Fig.2).

2. To find the solution $\psi(\theta,\eta)$ of equation (1.2) in the triangle $abc(adc)$, which becomes zero on the line $\theta = \mu$ and matches the solution of the preceding problem on the characteristic $\eta = \alpha' (\xi = \alpha)$. The constants above have the following definitions.

Flow pattern I — lower (upper) rear face of profile:

$$\alpha = 0, \quad \beta = \theta_1, \quad \mu = 1, \quad \alpha' = 2 \quad (\alpha' = 1, \quad \beta' = \theta_1, \quad \mu = 0, \quad \alpha = -1) \tag{1.4}$$

Flow pattern II — lower rear (upper) front face of profile

$$\alpha = 0, \quad \beta = \theta_1, \quad \mu = \frac{2\theta_0}{\pi}, \quad \alpha' = \frac{4\theta_0}{\pi}$$
$$\left(\alpha' = 1, \quad \beta' = \theta_1, \quad \mu = \frac{2\theta_0}{\pi}, \quad \alpha = -1 + \frac{2\theta_0}{\pi}\right) \tag{1.5}$$

Flow pattern III — upper surface of profile:

$$\alpha' = 1, \quad \beta' = \theta_1, \quad \mu = 0, \quad \alpha = -1 \tag{1.6}$$

Here, in all three cases, $\theta_1 = -al$, is the value of θ which corresponds to the undisturbed free stream.

In [2], the solution of the first problem was found, which takes on zero value on the characteristic $\eta = \alpha'$:

$$u(\xi, \lambda) = \gamma \int_{\lambda}^{\alpha'} \frac{\nu(t)\,dt}{(t-\lambda)^{1/6}(t-\xi)^{1/6}} \qquad \left(\gamma = \frac{3^{2/3}\Gamma^3(1/3)}{4\pi^2}\right) \tag{1.7}$$

In the same manner, we find that the solution of the first problem, which takes on the zero value on the characteristic $\xi = \alpha$ has the form

$$u(\xi, \lambda) = \gamma \int_{\alpha}^{\xi} \frac{\nu(t)\,dt}{(\xi - t)^{1/6}(\lambda - t)^{1/6}} \tag{1.8}$$

The solution of the second problem in the triangle abc can now be written as a simple linear combination of (1.7) and (1.8):

$$\psi(\xi, \lambda) = \gamma \int_{\alpha}^{\xi} \frac{\nu(t)\,dt}{(\xi - t)^{1/6}(\lambda - t)^{1/6}} - \gamma \int_{\lambda}^{2\mu-\alpha} \frac{\nu(2\mu - t)\,dt)}{(t-\lambda)^{1/6}(t-\xi)^{1/6}} \tag{1.9}$$

Indeed, if $\lambda = \alpha' = 2\mu - \alpha$ it is seen that (1.9) matches (1.7) on the characteristic $\lambda = \alpha'$. Furthermore, if $\theta = \mu$, we find from (1.3) that

$\xi = 2\mu - \lambda$; then, with the change of variable of integration $t' = 2\mu - t$ in the second of integrals in (1.9), we conclude that $[\psi(\xi,\lambda)]_{\theta=\mu} = 0$.

Similarly, the solution of the second problem in the triangle adc takes the form:

$$\psi(\xi,\lambda) = \gamma \int\limits_{\lambda}^{\alpha_1} \frac{\nu(t)\,dt}{(t-\xi)^{1/6}(t-\lambda)^{1/6}} - \gamma \int\limits_{2\mu-\alpha'}^{\xi} \frac{\nu(2\mu-t)\,dt}{(\xi-t)^{1/6}(\lambda-t)^{1/6}}, \tag{1.10}$$

2. Return to the physical plane

The return to the physical plane rests on the determination of the function $\phi(\theta,\eta)$, which corresponds to the velocity potential and appears as the dimensionless coordinate x/l_* in that plane. For a given function $\psi(\theta,\eta)$, which satisfies (1.2), the function $\phi(\theta,\eta)$ is determined from equations (1.1) up to an additive constant.

In order to find the pressure distribution over the airfoil one needs the values of ϕ on the lines $\theta = \mu$ which correspond to one or the other face of the profile, depending upon the choice of μ. These values of ϕ can be computed as follows. Taking account of (1.1) and the fact that $\alpha\phi = (\partial\phi/\partial\lambda)\,d\eta$ along the line $\theta = \mu$, we have:

$$\varphi(\mu,\eta) = C + \int\limits_{\eta_c}^{\eta} \varphi_\eta(\mu,\eta)\,d\eta = C + \int\limits_{\eta_c}^{\eta} \eta\psi_\theta(\mu,\eta)\,d\eta \tag{2.1}$$

where C is the value of ϕ at the point c and y_c the ordinate of that point. Furthermore, from (1.3) follows that $\partial\psi/\partial\theta = \partial\psi/\partial\xi + \partial\psi/\partial\lambda$ and equation (2.1) may be written either

$$\varphi(\mu,\eta) = C - \left(\frac{3}{2}\right)^{1/3} \int\limits_{\alpha}^{\xi} (\mu - t)^{1/3}\,[\psi_\xi(t, 2\mu - t) + \psi_\lambda(t, 2\mu - t)]\,dt \tag{2.2}$$

or

$$\varphi(\mu,\eta) = C - \left(\frac{3}{2}\right)^{1/3} \int\limits_{\lambda}^{\alpha'} (t - \mu)^{1/3}\,[\psi_\xi(2\mu - t, t) + \psi_\lambda(2\mu - t, t)]\,dt \tag{2.3}$$

Keeping in mind that along the line $\xi + \lambda = 2\mu$ the identity

$$\frac{\partial}{\partial\xi} f(\xi,\lambda) = \frac{\partial}{\partial\xi} f(\xi,\lambda) + \frac{\partial}{\partial\lambda} f(\xi,\lambda)$$

applies, and that also

$$\frac{df(\lambda)}{d\lambda} = -\frac{df(2\mu-\xi)}{d\xi}$$

we arrive at the function $\psi(\xi,\lambda)$, determined from (1.9) after some simple transformations:

$$[\psi_\xi(2\mu-\lambda,\lambda) + \psi_\lambda(2\mu-\lambda,\lambda)] = \gamma\left[\frac{2}{3}\int\limits_{\lambda}^{\alpha'} \frac{\nu(t)\,dt}{(t-\lambda)^{1/6}(t-2\mu-\lambda)^{7/6}} + \right.$$

$$\left. + 2\frac{d}{d\lambda}\int\limits_{\lambda}^{\alpha'} \frac{\nu(t)\,dt}{(t-2\mu+\lambda)^{1/6}(t-\lambda)^{1/6}}\right] \tag{2.4}$$

Substituting (2.4) into (2.3), integrating by parts, changing the order of integration according to Dirichlet's formula, and regrouping, we find:

$$\varphi(\mu,\eta) = C + \gamma\left(\frac{3}{2}\right)^{1/3}\int_{\lambda}^{\alpha'} \nu(t)\,F\left(\frac{t-\lambda}{t-\mu}\right)dt \tag{2.5}$$

where

$$F(x) = 2\frac{(1-x)^{1/3}}{(2-x)^{1/6}x^{1/6}} + \frac{2}{3}x^{5/6}\int_0^1 t^{-1/6}(1-xt)^{-2/3}(2-xt)^{-7/6}\,dt$$

The last integral may be expressed in terms of a hypergeometric function so that the result takes the form:

$$F(x) = (1-x)^{-2/3}\,\Phi_1[x(2-x)] \qquad \left(\Phi_1(x) = x^{-1/6}\left[2 - \frac{4}{5}xF\left(1, \frac{1}{2}, \frac{11}{6}, x\right)\right]\right)$$

The formula (2.5) finally is written:

$$\varphi(\mu,\eta) = C + \gamma\left(\frac{3}{2}\right)^{1/3}(\mu-\xi)^{-2/3}\int_{\alpha}^{\xi}\nu(t)(\mu-t)^{2/3}\,\Phi_1\left[1-\left(\frac{\mu-\xi}{\mu-t}\right)^2\right]dt \tag{2.6}$$

The resultant expression represents $\phi\ (\mu,\eta)$ as an integral operator of $\nu(t)$, prescribed on the interval $\mu < t \leqslant \alpha'$. A similar formula can be written for the interval $\alpha \leqslant t < \mu$ utilizing (2.2):

$$\varphi(\mu,\eta) = C + \gamma\left(\frac{3}{2}\right)^{1/3}(\mu-\xi)^{-2/3}\int_{\alpha}^{\xi}\nu(t)(\mu-t)^{2/3}\,\Phi_1\left[1-\left(\frac{\mu-\xi}{\mu-t}\right)^2\right]dt \tag{2.7}$$

In the case where the derivative of $\nu(t)$ exists over the interval of its definition, the above expressions may be transformed by integration by parts into

$$\varphi(\mu,\eta) = C - \frac{6}{5}\left(\frac{3}{2}\right)^{1/3}\gamma\int_{\lambda}^{\alpha'}\nu'(t)\,\Phi\left[1-\left(\frac{\lambda-\mu}{t-\mu}\right)^2\right](t-\mu)\,dt \tag{2.8}$$

$$\varphi(\mu,\eta) = C - \frac{6}{5}\left(\frac{3}{2}\right)^{1/3}\gamma\int_{\alpha}^{\xi}\nu'(t)\,\Phi\left[1-\left(\frac{\mu-\xi}{\mu-t}\right)^2\right](\mu-t)\,dt \tag{2.9}$$

where

$$\Phi(x) = x^{5/6}F\left(\frac{5}{6}, \frac{1}{3}, \frac{11}{6}, x\right), \qquad \lambda = \mu + \frac{2}{3}(-\eta)^{3/2}, \qquad \xi = \mu - \frac{2}{3}(-\eta)^{3/2}$$

These last expressions are more convenient for computations because the function $\Phi\ (x)$ is very nearly linear, namely $y = 1.26\,x$.

In the cases of interest, one may set $\alpha = 0$, $\alpha' = 1$ in the formulas (2.8) and (2.9).

3. Pressure distributions and limitations of the flow approximations

The dimensionless coefficient of pressure is easily expressed through the variable η

$$\bar{p} = 2^{5/3} \frac{\theta_0^{5/3}}{(\varkappa + 1)^{1/2}} a^{-2/3} \eta \tag{3.1}$$

In the Flow pattern I, we have $\mu = 0$ for the upper front face and $\mu = 1$ for the lower rear face. Since in the upper corner, $\phi = 1$, then $C = 1$ and therefore the expressions (2.8) or (2.9) take the form

$$\varphi(0, \eta) = 1 - \frac{6}{5}\left(\frac{3}{2}\right)^{1/3} \gamma \int_{\delta}^{1} \nu'(t)\, \Phi\left(1 - \frac{\delta^2}{t^2}\right) t\, dt \tag{3.2}$$

$$\delta = \frac{2}{3}(-\eta)^{3/2}$$

$$\varphi(1, \eta) = 1 - \frac{6}{5}\left(\frac{3}{2}\right)^{1/3} \gamma \int_{0}^{1-\delta} \nu'(t)\, \Phi\left(1 - \frac{\delta}{(1-t)^2}\right)(1-t)\, dt \tag{3.3}$$

In [5], the expression for $\nu(0)$ was derived. Differentiating $\nu(\theta)$ and substituting into (3.2) or (3.3), one can obtain the pressure distribution over the rear faces of a double-wedge profile under the conditions of Flow pattern I. Fig.3 displays the results of computations of η and $d\eta/d\theta_1$ for the rear face according to formulas (3.2), (3.3). The dashed line indicates the results of [1].

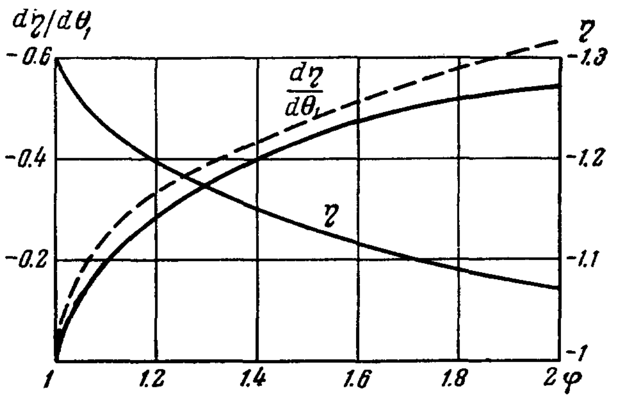

Fig. 3.

Numerical integration of the resulting pressure distribution yields, for Flow pattern I,

$$C_x = 5.30\, \frac{\theta_0^{5/3}}{(\varkappa + 1)^{1/3}}$$

$$C_y = \frac{3.21}{[\theta_0 (\varkappa + 1)]^{1/3}}\, \alpha$$

$$C_m = \frac{0.911}{[\theta_0 (\gamma + 1)]^{1/3}}\, \alpha$$

These formulas are applicable for sufficiently small ratios α/θ_0.

In case of Flow pattern II over the upper front face, $\phi = 0$ at the leading edge. Then, setting $\mu = a^1$, $C = 0$ we have:

$$\varphi(a, \eta) = -\frac{6}{5}\left(\frac{3}{2}\right)^{1/3} \gamma \int\limits_{\alpha+\delta}^{1} \nu'(t)\, \Phi\left[1 - \frac{\delta^2}{(1-t)^2}\right](t-a)\, dt \tag{3.4}$$

Starting from the study of the flow over the lower front face under conditions of Flow pattern II when θ and θ_1 are sufficiently small, we have [5]:

$$\nu(\theta) = -\frac{7}{18}\frac{1}{a^2} f(l, t) \tag{3.5}$$

where

$$f(l, t) = l^{4/3}\left[(t+l)^{-10/3} + (t-l)^{-10/3}\right] - \frac{3}{7} l^{1/3}\left[(t-l)^{-7/3} - (t+l)^{-7/3}\right]$$

$$\theta = at, \qquad \theta_1 = -al, \qquad l = -\frac{\alpha+\theta_0}{2\theta_0}, \qquad a = \frac{2\theta_0}{\pi}$$

Substituting (3.5) into (3.4), changing the variable of integration $t = at$ in the latter, and going to the limit as $a \to 0$ ($\theta_0 \to 0$), we obtain:

$$\varphi(0, \eta) = \frac{7}{15}\left(\frac{3}{2}\right)^{1/3} \gamma \int\limits_{1+\rho}^{\infty} f(l, t)\, \Phi\left[1 - \frac{\rho^2}{(t-1)^2}\right](t-1)\, dt \tag{3.6}$$

Here

$$\rho = \delta / a = \pi / 3 = \left(-\eta / \theta_0^{2/3}\right)^{3/2}$$

Along the lower rear surface of the profile $\theta = a$ and the coordinate of the lower corner will be $\phi = 1$.

Setting $\mu = a$; $C = 1$ in (3.5) and (2.9) and proceeding to the limit as $a \to 0$:

$$\varphi(0, \eta) = 1 - \frac{7}{15}\left(\frac{3}{2}\right)^{1/3} \gamma \int\limits_{0}^{1-\rho} f(l, t)\, \Phi\left[1 - \frac{\rho^2}{(1-t)^2}\right](1-t)\, dt \tag{3.7}$$

Figs. 4 and 5 show the dependence of $\eta\ \theta_0^{2/3}$ on ϕ for four values of the parameter $\overline{\alpha} = \alpha/\theta_0$, calculated by formulas (3.6) and (3.7), which represent the pressure distribution on the upper front and lower rear faces of the profile.

The formulas allow the assessment of the bounds on α/θ_0 within which the Flow pattern II represents the flow realistically. Clearly, Flow pattern II obtains when there is no sonic point on the upper front and lower rear faces. In other cases, the flow is recomputed with representation of local supersonic region near the nose or the lower corner of the profile.

Setting $\rho = 0$ ($\eta = 0$) in (3.6) or (3.7) and carrying out the integration we obtain the coordinates of the sonic point on the upper front face:

$$\varphi_+ = \frac{1}{4}\, l^{4/3}\left[(1+l)^{-4/3} + (1-l)^{-4/3}\right] - \frac{3}{4}\left[(1-l)^{-1/3} - (1+l)^{-1/3}\right] \tag{3.8}$$

and on the lower rear face

$$\varphi_- = \varphi_+ - 1 \tag{3.9}$$

Graphically we discover that $\phi_+ \geqslant 1$ and $\phi_- \geqslant 2$ occur when

$$0.59 \leqslant \frac{\alpha}{\theta_0} \leqslant 1.94 \tag{3.10}$$

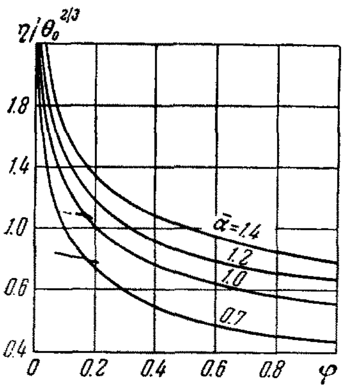

Fig. 4.

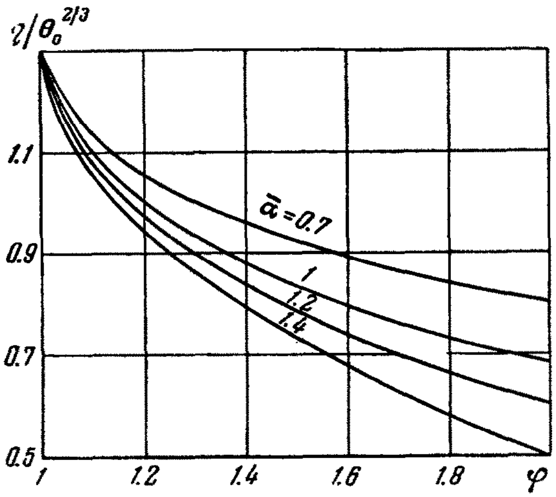

Fig. 5.

The last inequality determines the bounds of applicability of Flow pattern II. In [5] it was experimentally established that the flow pattern II is realized for $\alpha/\theta_0 > 0.6$ which agrees well with (3.10).

BIBLIOGRAPHY

1. Guderley, G. and Yoshihara, H., Two dimensional asymmetric flow patterns at Mach number unity. *J. Aero. Sci.* Vol. 20, No. 11, 1953.

2. Kriuchin, A.F., O soprotivlenii rombovidnogo profilia pri okolzvukovykh skorostiakh (On the drag of rhomboidal profiles near sonic speeds). *Dokl. Akad. Nauk SSSR*, Vol. 97, No. 2, 1954.

3. Guderley, G., The flow over a flat plate with a smallangle of attack at Mach number unity. *J. Aero. Sci.* Vol. 21, No. 4, 1954.

4. Skripkin, V.A., Ob okolzvukovom istechenii gazovoi strui iz nasadka parallel'nymi stenkami (On the transonic flow of a gas jet out of a nozzle with parallel walls). *PMM* Vol. 19, No. 1, 1955.

5. Kopylov, G.N., Zvukovoi protok okolo klina pod nekotorym uglom ataki (Sonic flow around a wedge at an angle of attack). *PMM* Vol. 21, No. 1, 1957.

6. Ovsianikov, L.V., Uravnenie okolzvukovogo dvizhenia gaza (Equations of transonic flows of gas). *Vest. LGU, ser. mekh. fiz. khim.* No. 6, 1952.

Translated by M.V.M.